Algebra 3 Mid-semester examination

There are two parts to this test. **Morning:** **10 am – 1 pm, out of 45 points**
Afternoon: **1 pm – 3 pm, out of 15 points**

You must submit your "cheat sheet" at the end of each part. Unless otherwise stated, the following are true: All rings are commutative with a multiplicative identity 1 that is different from 0. All ideals are proper.

VERY IMPORTANT: Write your answers precisely and justify everything. You may use any results proved in the book (by making clear which result you are using), but no others.

1. [7 points] Find all automorphisms of **Z**[X]. Conclude that given a fixed integer c, every element of **Z**[X] can be represented uniquely as a polynomial in (x – c) with integer coefficients.

2. [7 points] Ideals I and J in a ring R satisfy I + J = R.

a) Prove that I ∩ J = I J
b) Prove that R / I J is isomorphic to R/I × R/J. Find the idempotents in R / I J corresponding this product decomposition.

3. [7 points] a) Show that an ideal P in a ring R is prime if and only if R/P is an integral domain.
b) Let *f*: R → D be a ring homomorphism into an integral domain D. Given two ideals in D, let I and J be their inverse images under *f* in R. Suppose the product ideal I J is contained in ker(*f*). Show that I or J equals ker(*f*). Is it necessary that I J equals ker(*f*)?

4. [8 points] For an element r in a ring R, consider the ideal I = (rX – 1) in R[X]. Consider the natural homomorphism Φ: R → S = R[X] / I. (In other words, Φ is the inclusion of R into R[X] followed by the natural surjection of R[X] onto S.)

a) Show that ker(Φ) is $\{a \mid r^n a = 0 \text{ for some } n > 0\}$.
b) Conclude that S = 0 if and only if r is nilpotent in R.
c) Show that Φ is an isomorphism if and only if r is a unit in R.

5. [8 points] Let M be a proper ideal in a ring R.

a) Show that the statement "All elements of R – M are units" is equivalent to the statement "M is the unique maximal ideal of R"
b) Using your knowledge of units in the power series ring **Q**[[X]], state why the equivalent conditions in part a hold for this ring.
c) Show that there is a unique homomorphism **Q**[[X]] → **Q**. Is this statement true if we replace **Q** by an arbitrary field F (same F in both places)?

6. [8 points] Solve the following independent problems concerning gcd.

a) Let R be a PID and S a UFD, with R contained in S. Let d = gcd(a,b) in R, where a and b are two nonzero nonunits in R. Show that d is also a gcd of a and b in S.

b) Find a gcd of (11+7i) and (18-i) in the ring of Gaussian integers **Z**[i].

Algebra 3 Mid-semester examination part 2

Solve problems worth at least 15 points (to get to a total of 60). Solving more may earn extra credit!

Now you may quote any result proved in the textbook *or* from the exercises in the textbook. (Of course, if the exam problem itself is from the exercises in the text, then you cannot quote it!)

As before: Write your answers precisely and justify everything. You must submit your "cheat sheet" at the end. Unless otherwise stated, you may assume that all rings are commutative with a multiplicative identity 1 that is different from 0 and that all ideals are proper.

7. [4+4 = 8 points] Given two polynomials f and g in $\mathbf{C}[X,Y]$, let $I = (f,g)$, the ideal generated by f and g. Prove that $\mathbf{C}[X,Y] / I$ is a finite dimensional vector space over $\mathbf{C}$ if and only if $\gcd(f,g) = 1$.

8. [4 points] Let R be a commutative ring. Describe the kernel of the map $\varphi: R[X,Y] \to R[T]$, where φ is identity on R, $\varphi(X) = T^p$ and $\varphi(Y) = T^q$. Here p and q are relatively prime positive integers.

9. [2+2+2 = 6 points] Solve the following independent questions.

a) An element r in a ring R of characteristic 5 satisfies $r^{999} = 0$. Find an $n>0$ such that $(1 + r)^n = 1$.

b) Let F = the two element ring. Find a reducible polynomial in F[X] of smallest possible positive degree that has no root in F.

c) Find all monic polynomials g(X) in $\mathbf{Q}[X]$ such that whenever f(X) is irreducible, so is f(g(X)).

10. [12 points] Show that up to isomorphism, there are exactly 4 rings of cardinality 4. What about rings of cardinality 9? I am looking for a *proof* here, so a mere listing will not get much credit.